4-6 Cramer's Rule

Main Ideas

- Solve systems of two linear equations by using Cramer's Rule.
- Solve systems of three linear equations by using Cramer's Rule.

New Vocabulary

Cramer's Rule

GET READY for the Lesson

Two sides of a triangle are contained in lines whose equations are $1.4x + 3.8y = 3.4$ and $2.5x - 1.7y = -10.9$. To find the coordinates of the vertex of the triangle between these two sides, you must solve the system of equations. One method for solving systems of equations is Cramer's Rule.

Systems of Two Linear Equations **Cramer's Rule** uses determinants to solve systems of equations. Consider the following system.

$ax + by = e$ — *a*, *b*, *c*, *d*, *e*, and *f* represent constants, *not* variables.

$cx + dy = f$

Solve for x by using elimination.

$adx + bdy = de$	Multiply the first equation by *d*.
$(-)\ bcx + bdy = bf$	Multiply the second equation by *b*.
$adx - bcx = de - bf$	Subtract.
$(ad - bc)x = de - bf$	Factor.
$x = \dfrac{de - bf}{ad - bc}$	Divide. Notice that $ad - bc$ must not be zero.

Solving for y in the same way produces the following expression.

$$y = \frac{af - ce}{ad - bc}$$

So the solution of the system of equations is $\left(\dfrac{de - bf}{ad - bc}, \dfrac{af - ce}{ad - bc}\right)$.

The fractions have a common denominator. It can be written using a determinant. The numerators can also be written as determinants.

$$ad - bc = \begin{vmatrix} a & b \\ c & d \end{vmatrix} \qquad de - bf = \begin{vmatrix} e & b \\ f & d \end{vmatrix} \qquad af - ce = \begin{vmatrix} a & e \\ c & f \end{vmatrix}$$

Study Tip

Look Back

To review **solving systems of equations**, see Lesson 3-2.

KEY CONCEPT — Cramer's Rule for Two Variables

The solution of the system of linear equations

$ax + by = e$
$cx + dy = f$

is (x, y), where $x = \dfrac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}$, $y = \dfrac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}$, and $\begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0$.

EXAMPLE System of Two Equations

1 **Use Cramer's Rule to solve the system of equations.**

$5x + 7y = 13$

$2x - 5y = 13$

$$x = \frac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} \quad \text{Cramer's Rule} \quad y = \frac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}$$

$$= \frac{\begin{vmatrix} 13 & 7 \\ 13 & -5 \end{vmatrix}}{\begin{vmatrix} 5 & 7 \\ 2 & -5 \end{vmatrix}} \quad a = 5, b = 7, c = 2, d = -5, e = 13, \text{ and } f = 13 \quad = \frac{\begin{vmatrix} 5 & 13 \\ 2 & 13 \end{vmatrix}}{\begin{vmatrix} 5 & 7 \\ 2 & -5 \end{vmatrix}}$$

$$= \frac{13(-5) - 13(7)}{5(-5) - 2(7)} \quad \text{Evaluate each determinant.} \quad = \frac{5(13) - 2(13)}{5(-5) - 2(7)}$$

$$= \frac{-156}{-39} \text{ or } 4 \quad \text{Simplify.} \quad = \frac{39}{-39} \text{ or } -1$$

The solution is $(4, -1)$.

CHECK Your Progress

Use Cramer's Rule to solve the systems of equations.

1A. $4x - 2y = -2$
$-x + 3y = 13$

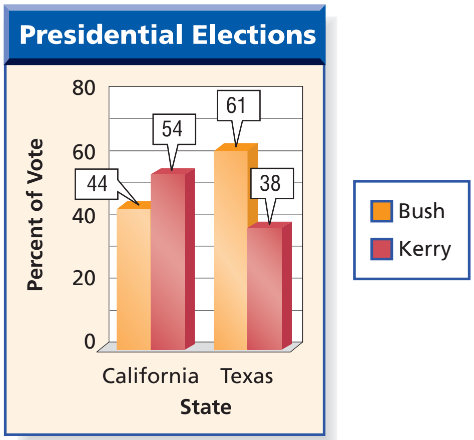
1B. $2x - 3y = 12$
$-6x + y = -20$

Real-World Link

In 2000, George W. Bush became the first son of a former president to win the presidency since John Quincy Adams did it in 1825.

Real-World EXAMPLE

2 **ELECTIONS** **In the 2004 presidential election, George W. Bush received about 10,000,000 votes in California and Texas, while John Kerry received about 9,500,000 votes in those states. The graph shows the percent of the popular vote that each candidate received in those states.**



a. **Write a system of equations that represents the total number of votes cast for each candidate in these two states.**

Words George W. Bush received 44% and 61% of the **votes in California** and **Texas**, respectively, for a total of 10,000,000 votes.

John Kerry received 54% and 38% of the **votes in California** and **Texas**, respectively, for a total of 9,500,000 votes.

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You know the total votes for each candidate in Texas and California and the percent of the votes cast for each. You need to know the number of votes for each candidate in each state.

Variables Let x represent the total number of votes in California.
Let y represent the total number of votes in Texas.

Equations $0.44x + 0.61y = 10{,}000{,}000$ Votes for Bush
$0.54x + 0.38y = 9{,}500{,}000$ Votes for Kerry

b. Find the total number of popular votes cast in California and Texas.
Use Cramer's Rule to solve the system of equations.
Let $a = 0.44$, $b = 0.61$, $c = 0.54$, $d = 0.38$, $e = 10{,}000{,}000$, and $f = 9{,}500{,}000$.

$$x = \frac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} \qquad \text{Cramer's Rule} \qquad y = \frac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}$$

$$= \frac{\begin{vmatrix} 10{,}000{,}000 & 0.61 \\ 9{,}500{,}000 & 0.38 \end{vmatrix}}{\begin{vmatrix} 0.44 & 0.61 \\ 0.54 & 0.38 \end{vmatrix}} \qquad = \frac{\begin{vmatrix} 0.44 & 10{,}000{,}000 \\ 0.54 & 9{,}500{,}000 \end{vmatrix}}{\begin{vmatrix} 0.44 & 0.61 \\ 0.54 & 0.38 \end{vmatrix}}$$

$$= \frac{10{,}000{,}000(0.38) - 9{,}500{,}000(0.61)}{0.44(0.38) - 0.54(0.61)} \qquad = \frac{0.44(9{,}500{,}000) - 0.54(10{,}000{,}000)}{0.44(0.38) - 0.54(0.61)}$$

$$= \frac{-1995000}{-0.1622} \qquad = \frac{-1220000}{-0.1622}$$

$$\approx 12{,}299{,}630 \qquad \approx 7{,}521{,}578$$

The solution of the system is about (12,299,630, 7,521,578).

So, there were about 12,300,000 popular votes cast in California and about 7,500,000 popular votes cast in Texas.

CHECK If you add the votes that Bush and Kerry received, the result is 10,000,000 + 9,500,000 or 19,500,000. If you add the popular votes in California and Texas, the result is 12,300,000 + 7,500,000 or 19,800,000. The difference of 300,000 votes is reasonable considering there were over 19 million total votes.

CHECK Your Progress

At the game on Friday, the Athletic Boosters sold chips *C* for \$0.50 and candy bars *B* for \$0.50 and made \$27. At Saturday's game, they raised the prices of chips to \$0.75 and candy bars to \$1.00. They made \$48 for the same amount of chips and candy bars sold.

2A. Write a system of equations that represents the total number of chips and candy bars sold at the games on Friday and Saturday.

2B. Find the total number of chips and candy bars that were sold on each day.

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Systems of Three Linear Equations You can also use Cramer's Rule to solve a system of three equations in three variables.

KEY CONCEPT *Cramer's Rule for Three Variables*

The solution of the system whose equations are

$$ax + by + cz = j$$
$$dx + ey + fz = k$$
$$gx + hy + iz = \ell$$

is (x, y, z), where $x = \frac{\begin{vmatrix} j & b & c \\ k & e & f \\ \ell & h & i \end{vmatrix}}{\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}}$, $y = \frac{\begin{vmatrix} a & j & c \\ d & k & f \\ g & \ell & i \end{vmatrix}}{\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}}$, $z = \frac{\begin{vmatrix} a & b & j \\ d & e & k \\ g & h & \ell \end{vmatrix}}{\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}}$, and $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} \neq 0$.

EXAMPLE System of Three Equations

3 **Use Cramer's Rule to solve the system of equations.**

$$3x + y + z = -1$$
$$-6x + 5y + 3z = -9$$
$$9x - 2y - z = 5$$

Cross-Curricular Project

Math Online You can use Cramer's Rule to compare home loans. Visit algebra2.com to continue work on your project.

$$x = \frac{\begin{vmatrix} j & b & c \\ k & e & f \\ \ell & h & i \end{vmatrix}}{\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}} \qquad y = \frac{\begin{vmatrix} a & j & c \\ d & k & f \\ g & \ell & i \end{vmatrix}}{\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}} \qquad z = \frac{\begin{vmatrix} a & b & j \\ d & e & k \\ g & h & \ell \end{vmatrix}}{\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}}$$

$$= \frac{\begin{vmatrix} -1 & 1 & 1 \\ -9 & 5 & 3 \\ 5 & -2 & -1 \end{vmatrix}}{\begin{vmatrix} 3 & 1 & 1 \\ -6 & 5 & 3 \\ 9 & -2 & -1 \end{vmatrix}} \qquad = \frac{\begin{vmatrix} 3 & -1 & 1 \\ -6 & -9 & 3 \\ 9 & 5 & -1 \end{vmatrix}}{\begin{vmatrix} 3 & 1 & 1 \\ -6 & 5 & 3 \\ 9 & -2 & -1 \end{vmatrix}} \qquad = \frac{\begin{vmatrix} 3 & 1 & -1 \\ -6 & 5 & -9 \\ 9 & -2 & 5 \end{vmatrix}}{\begin{vmatrix} 3 & 1 & 1 \\ -6 & 5 & 3 \\ 9 & -2 & -1 \end{vmatrix}}$$

Use a calculator to evaluate each determinant.

$x = \frac{-2}{-9}$ or $\frac{2}{9}$ $\qquad$ $y = \frac{12}{-9}$ or $-\frac{4}{3}$ $\qquad$ $z = \frac{3}{-9}$ or $-\frac{1}{3}$

The solution is $\left(\frac{2}{9}, -\frac{4}{3}, -\frac{1}{3}\right)$.

CHECK Your Progress

3. $2x + y - z = -2$
$-x + 2y + z = -0.5$
$x + y + 2z = 3.5$

✓ CHECK Your Understanding

Example 1 (p. 202)

Use Cramer's Rule to solve each system of equations.

1. $x - 4y = 1$
$2x + 3y = 13$

2. $0.2a = 0.3b$
$0.4a - 0.2b = 0.2$

Example 2 (pp. 202–203)

INVESTING For Exercises 3 and 4, use the following information.
Jarrod Wright has a total of \$5000 in his savings account and in a certificate of deposit. His savings account earns 3.5% interest annually. The certificate of deposit pays 5% interest annually if the money is invested for one year. He calculates that his interest earnings for the year will be \$227.50.

3. Write a system of equations for the amount of money in each investment.

4. How much money is in his savings account and in the certificate of deposit?

Example 3 (p. 204)

Use Cramer's Rule to solve each system of equations.

5. $2x - y + 3z = 5$
$3x + 2y - 5z = 4$
$x - 4y + 11z = 3$

6. $a + 9b - 2c = 2$
$-a - 3b + 4c = 1$
$2a + 3b - 6c = -5$

Exercises

HOMEWORK HELP

For Exercises	See Examples
7–12	1
13–17	2
18–21	3

Use Cramer's Rule to solve each system of equations.

7. $5x + 2y = 8$
$2x - 3y = 7$

8. $2m + 7n = 4$
$m - 2n = -20$

9. $2r - s = 1$
$3r + 2s = 19$

10. $3a + 5b = 33$
$5a + 7b = 51$

11. $2m - 4n = -1$
$3n - 4m = -5$

12. $4x + 3y = 6$
$8x - y = -9$

13. GEOMETRY The two sides of an angle are contained in lines whose equations are $4x + y = -4$ and $2x - 3y = -9$. Find the coordinates of the vertex of the angle.

14. GEOMETRY Two sides of a parallelogram are contained in the lines whose equations are $2.3x + 1.2y = 2.1$ and $4.1x - 0.5y = 14.3$. Find the coordinates of a vertex of the parallelogram.

STATE FAIR For Exercises 15 and 16, use the following information.
Jackson and Drew each purchased some game and ride tickets.

Person	Ticket Type	Tickets	Total
Jackson	game ride	6 15	\$93
Drew	game ride	7 12	\$81

15. Write a system of two equations using the given information.

16. Find the price for each type of ticket.

17. RINGTONES Ella's cell phone provider sells standard and premium ringtones. One month, Ella bought 2 standard and 2 premium ringtones for \$8.96. The next month Ella paid \$9.46 for 1 standard and 3 premium ringtones. What are the prices for standard and premium ringtones?

Use Cramer's Rule to solve each system of equations.

18. $x + y + z = 6$
$2x + y - 4z = -15$
$5x - 3y + z = -10$

19. $a - 2b + c = 7$
$6a + 2b - 2c = 4$
$4a + 6b + 4c = 14$

20. $r - 2s - 5t = -1$
$r + 2s - 2t = 5$
$4r + s + t = -1$

21. $3a + c = 23$
$4a + 7b - 2c = -22$
$8a - b - c = 34$

22. $4x + 2y - 3z = -32$
$-x - 3y + z = 54$
$2y + 8z = 78$

23. $2r + 25s = 40$
$10r + 12s + 6t = -2$
$36r - 25s + 50t = -10$

24. $0.5r - s = -1$
$0.75r + 0.5s = -0.25$

25. $1.5m - 0.7n = 0.5$
$2.2m - 0.6n = -7.4$

26. $\frac{1}{3}r + \frac{2}{5}s = 5$
$\frac{2}{3}r - \frac{1}{2}s = -3$

27. $\frac{3}{4}x + \frac{1}{2}y = \frac{11}{12}$
$\frac{1}{2}x - \frac{1}{4}y = \frac{1}{8}$

Real-World Link

Video games are becoming increasingly popular among adults. In fact, more than 5% of adults play video games 2 or more times per week.

Source: U.S. Census Bureau

28. ARCADE GAMES Marcus and Cody purchased game cards to play virtual games at the arcade. Marcus used 47 points from his game card to drive the race car simulator and the snowboard simulator four times each. Cody used 48.25 points from his game card to drive the race car five times and the snowboard three times. How many points does each game charge per play?

29. PRICING The Harvest Nut Company sells made-to-order trail mixes. Sam's favorite mix contains peanuts, raisins, and carob-coated pretzels. Peanuts sell for $3.20 per pound, raisins are $2.40 per pound, and the carob-coated pretzels are $4.00 per pound. Sam bought a 5-pound mixture for $16.80 that contained twice as many pounds of carob-coated pretzels as raisins. How many pounds of peanuts, raisins, and carob-coated pretzels did Sam buy?

EXTRA PRACTICE
See pages 898, 929.
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H.O.T. Problems

30. OPEN ENDED Write a system of equations that *cannot* be solved using Cramer's Rule.

31. REASONING Write a system of equations whose solution is

$$x = \frac{\begin{vmatrix} -6 & 5 \\ 30 & -2 \end{vmatrix}}{\begin{vmatrix} 3 & 5 \\ 4 & -2 \end{vmatrix}},\ y = \frac{\begin{vmatrix} 3 & -6 \\ 4 & 30 \end{vmatrix}}{\begin{vmatrix} 3 & 5 \\ 4 & -2 \end{vmatrix}}.$$

32. CHALLENGE In Cramer's Rule, if the value of the determinant is zero, what must be true of the graph of the system of equations represented by the determinant? Give examples to support your answer.

33. *Writing in Math* Use the information about two sides of the triangle on page 201 to explain how Cramer's Rule can be used to solve systems of equations. Include an explanation of how Cramer's rule uses determinants, and a situation where Cramer's rule would be easier to use to solve a system of equations than substitution or elimination.

STANDARDIZED TEST PRACTICE

34. ACT/SAT Each year at Capital High School the students vote to choose the theme of that year's homecoming dance. The theme "A Night Under the Stars" received 225 votes, and "The Time of My Life" received 480 votes. If 40% of girls voted for "A Night Under the Stars", 75% of boys voted for "The Time of My Life", and all of the students voted, how many girls and boys are there at Capital High School?

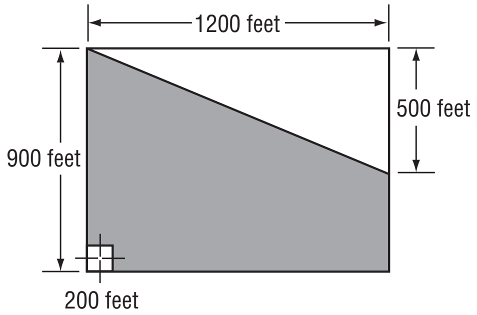
A 854 boys and 176 girls

B 705 boys and 325 girls

C 395 boys and 310 girls

D 380 boys and 325 girls

35. REVIEW What is the area of the shaded part of the rectangle below?



F 440,000 ft^2 **H** 640,000 ft^2

G 540,000 ft^2 **J** 740,000 ft^2

Spiral Review

Find the value of each determinant. (Lesson 4-5)

36. $\begin{vmatrix} 3 & 2 \\ -2 & 4 \end{vmatrix}$ **37.** $\begin{vmatrix} 8 & 6 \\ 4 & 8 \end{vmatrix}$ **38.** $\begin{vmatrix} -5 & 2 \\ 4 & 9 \end{vmatrix}$

For Exercises 39 and 40, use the following information. (Lesson 4-4)
Triangle ABC with vertices $A(0, 2)$, $B(-3, -1)$, and $C(-2, -4)$ is translated 1 unit right and 3 units up.

39. Write the translation matrix.

40. Find the coordinates of $\triangle A'B'C'$. Then graph the preimage and the image.

Solve each system of equations by graphing. (Lesson 3-1)

41. $y = 3x + 5$
$y = -2x - 5$

42. $x + y = 7$
$\frac{1}{2}x - y = -1$

43. $x - 2y = 10$
$2x - 4y = 12$

44. BUSINESS The Friendly Fix-It Company charges a base fee of \$45 for any in-home repair. In addition, the technician charges \$30 per hour. Write an equation for the cost c of an in-home repair of h hours. (Lesson 1-3)

GET READY for the Next Lesson

PREREQUISITE SKILL **Find each product, if possible.** (Lesson 4-3)

45. $\begin{bmatrix} 2 & 5 \end{bmatrix} \cdot \begin{bmatrix} 3 & 1 \\ -2 & 6 \end{bmatrix}$

46. $\begin{bmatrix} 0 & 9 \\ 5 & 7 \end{bmatrix} \cdot \begin{bmatrix} 2 & -6 \\ 8 & 1 \end{bmatrix}$

47. $\begin{bmatrix} 5 & -4 \\ 8 & 3 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 1 \end{bmatrix}$

48. $\begin{bmatrix} 7 & 11 & -5 \\ 3 & 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 8 \\ 5 \end{bmatrix}$